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Q.1 Define Continuity and differentiability with examples.

Ans: Continuity at a point:→ The function $f(x)$ is said to be continuous at $x=a$, if for a preassigned positive number ϵ, however small, a positive number δ exists such that $|f(x)-f(a)|<\epsilon$, provided that $|x-a|<\delta$.

The equivalence of the two definitions is at once obvious if the second be compared with the definition of limit.

Continuity in an interval:→ A function $f(x)$ is said to be continuous of the closed interval $[a,b]$, i.e. $a\le x\le b$, if it is continuous at every point of the open interval $]a,b[$, i.e. $a<x<b$, and if $f(a+0)$ exists and is equal to $f(a)$, and $f(b-0)$ exists and is equal to $f(b)$.

Differentiability:→ A function $f(x)$ defined at a point a and in a certain neighbourhood of a is said to be differentiable at a if the limits

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \text{ and } \lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}$$

both exist and have the same value.

The value is called the coefficient of $f(x)$ at a and is denoted by $f'(a)$.

If the limit in question only exists on the left or on the ~~right~~, then we speak of left hand or right hand differentiability, differential coefficient.

For example. $f(x) = 1+x$, if $x\le 2$, $f(x)=5-x$, if $x\ge 2$.

$f(x) = x\sin\frac{1}{x}$, for $x\neq 0$.

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Q.2. If a function is differentiable at a point, then prove that it must be continuous at that point.

Proof: Let f(x) be differentiable at $x=a$.

Then, by definition,

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}=A \text{ (say)}.$$

$\therefore |f(a+h)-f(a)|=|h||A+\epsilon|$, where $\epsilon\to 0$ as $h\to 0$

i.e. $f(a+h)-f(a)\to 0$ as $h\to 0$

i.e. $f(a+h)\to f(a)$ as $h\to 0$ ——— (I)

Also, $|f(a-h)-f(a)|=|h|[A+\epsilon_1]$, where $\epsilon_1\to 0$ as $h\to 0$

i.e. $f(a-h)-f(a)\to 0$ as $h\to 0$

i.e. $f(a-h)\to f(a)$ as $h\to 0$ ——— (II)

From (I) and (II), we get

$$\lim_{h\to 0} f(a+h)=f(a)=\lim_{h\to 0} f(a-h).$$

Hence, the function is continuous at $x=a$.

Q.3. Prove that the converse of this theorem is not necessarily true. (or, Show that continuity is a weaker condition on a function than differentiability.

Soln: Let us examine the continuity and the differentiability of the function $f(x)=|x|$ at $x=0$.

The function f(x) is said to be continuous at $x=0$ if

$$\lim_{h\to 0} f(0+h)=f(0)=\lim_{h\to 0} f(0-h)$$

Now, $\lim_{h\to 0} f(0+h)=\lim_{h\to 0}|0+h|=\lim_{h\to 0}|h|=\lim_{h\to 0} h=0.$

$f(0)=|0|=0$, $\lim_{h\to 0} f(0-h)=\lim_{h\to 0}|0-h|=\lim_{h\to 0}|-h|=\lim_{h\to 0} h=0$

$\therefore \lim_{h\to 0} f(0+h)=f(0)=\lim_{h\to 0} f(0-h)$

Hence, the function $f(x)=|x|$ is continuous at $x=0$.

Again, $\frac{f(x)-f(0)}{x}=\frac{|x|}{x}=\pm 1$, according as x is $\pm$ve.

Hence, $\lim_{x\to 0}\frac{f(x)-f(0)}{x}$ does not exist.

So, f(x) is not differentiable at $x=0$.

Thus, the function under consideration is continuous at $x=0$ but not differentiable at $x=0$.