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Q.1 Define Continuity and differentiability with examples.

Ans: Continuity at a point: →

The function $f(x)$ is said to be continuous at $x=a$, if for a preassigned positive number ϵ , however small, a positive number δ exists such that $|f(x) - f(a)| < \epsilon$, provided that $|x-a| < \delta$.

The equivalence of the two definitions is at once obvious if the second be compared with the definition of limit.

Continuity in an interval: → A function $f(x)$ is said to be continuous on the closed interval $[a, b]$, i.e. $a \leq x \leq b$, if it is continuous at every point of the open interval (a, b) , i.e. $a < x < b$, and if $f(a+0)$ exists and is equal to $f(a)$, and $f(b-0)$ exists and is equal to $f(b)$.

Differentiability: → A function $f(x)$ defined at a point a and in a certain neighbourhood of a is said to be differentiable at a if the limits

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \text{ and } \lim_{h \rightarrow 0} \frac{f(a-h) - f(a)}{-h}$$

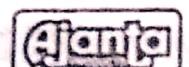
both exist and have the same value.

⊗ The value is called the coefficient of $f(x)$ at a and is denoted by $f'(a)$.

If the limit in question only exists on the left or on the right, then we speak of left hand or right hand differentiability, differential coefficient.

For example. $f(x) = 1+x$, if $x \leq 2$, $f(x) = 5-x$, if $x > 2$.

$$f(x) = \begin{cases} 1+x, & \text{for } x \neq 2 \\ 5-x, & \text{for } x=2 \end{cases}$$



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Q.2. If a function is differentiable at a point, then prove that it must be continuous at that point.

Proof: Let $f(x)$ be differentiable at $x=a$.

Then, by definition,

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{-h} = A / \text{say}$$

$$\therefore |f(a+h) - f(a)| = |h| |A + \epsilon|, \text{ where } \epsilon \rightarrow 0 \text{ as } h \rightarrow 0$$

i.e. $|f(a+h) - f(a)| \rightarrow 0$ as $h \rightarrow 0$

i.e. $f(a+h) \rightarrow f(a)$ as $h \rightarrow 0$

$$\text{Also, } |f(a+h) - f(a)| = |h| |A + \epsilon_1|, \text{ where } \epsilon_1 \rightarrow 0 \text{ as } h \rightarrow 0$$

i.e. $|f(a+h) - f(a)| \rightarrow 0$ as $h \rightarrow 0$

i.e. $f(a+h) \rightarrow f(a)$ as $h \rightarrow 0$

From I and II, we get

$$\lim_{h \rightarrow 0} f(a+h) = f(a) = \lim_{h \rightarrow 0} f(a-h)$$

Hence, the function is continuous at $x=a$.

Q.3. Prove that the converse of this theorem is not necessarily true. i.e., show that continuity is a weaker condition than differentiability.

Soln: Let us examine the continuity and the differentiability of the function $f(x) = |x|$ at $x=0$.

The function $f(x)$ is said to be continuous at $x=0$ if

$$\lim_{h \rightarrow 0} f(0+h) = f(0) = \lim_{h \rightarrow 0} f(0-h)$$

$$\text{Now, } \lim_{h \rightarrow 0} f(0+h) = \lim_{h \rightarrow 0} |0+h| = \lim_{h \rightarrow 0} |h| = \lim_{h \rightarrow 0} h = 0.$$

$$f(0) = |0| = 0, \lim_{h \rightarrow 0} f(0-h) = \lim_{h \rightarrow 0} |0-h| = \lim_{h \rightarrow 0} |-h| = \lim_{h \rightarrow 0} h = 0$$

$$\therefore \lim_{h \rightarrow 0} f(0+h) = f(0) = \lim_{h \rightarrow 0} f(0-h)$$

Hence, the function $f(x) = |x|$ is continuous at $x=0$.

Again, $\frac{f(x)-f(0)}{x} = \frac{|x|}{x} = \pm 1$, according as x is +ve.

Hence, $\lim_{x \rightarrow 0} \frac{f(x)-f(0)}{x}$ does not exist.

So, $f(x)$ is not differentiable at $x=0$.

Thus, the function under consideration is continuous at $x=0$ but not differentiable at $x=0$.